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Holomorphic discrete series and harmonic series unitary irreducible representations of non-compact Lie groups: Sp(2*n*, *R*), U(*p*, *q*) and SO*(2*n*)

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Abstract. Generalised characters of the infinite dimensional, holomorphic, discrete series, unitary, irreducible representations of the non-compact groups U(p, q), Sp($2n$, R) and SO*($2n$) are explicitly expressed in terms of characters of finite dimensional unitary group representations. These formulae are remarkably succinct despite involving certain infinite series of Schur functions. Similar formulae are derived for harmonic series unitary representation of both U(p, q) and Sp($2n$, R). Consideration of the branching rules from U(p, q) to U(q)×U(p) and from Sp($2n$, R) to U(n) enables holomorphic representations to be identified as a subset of the harmonic representations. The branching rules are established in full generality and are then used in the evaluation of tensor products of both holomorphic and harmonic representations. In the case of the former a known result is recast in terms of closed formulae involving Schur functions and for the latter various generalisations of these formulae are given. A conjecture is also made regarding what might be the simplest possible formulae covering all holomorphic and harmonic representations of Sp($2n$, R) and U(p, q). Illustrative examples are presented.

1. Introduction

Apart from rather trivial one-dimensional representations, all unitary irreducible representations (unirreps) of non-compact Lie groups are necessarily infinite dimensional. Despite this, such representations have a role to play in theoretical physics where they are encountered for example in the study of the harmonic oscillator (Hwa and Nuyts 1966, Moshinsky and Quesne 1971), the hydrogen atom (Barut and Kleinert 1967, Fronsdal 1967), the theory of nuclear collective motion (Arickx 1976, Rosensteel and Rowe 1977) and in various models of elementary particles and their interactions (Dothan *et al* 1965, Salam and Strathdee 1966).

Unfortunately a complete classification scheme for all such unitary irreducible representations of non-compact Lie groups is not yet available even in the case of simple Lie groups. However there exists a very extensive mathematical literature on the properties of both discrete series representations (Harish-Chandra 1956, 1966, Gelfand and Graev 1967, Graev 1968, Gelbart 1973, Schmid 1975, Hecht and Schmid 1975, Atiyah and Schmid 1977) and harmonic series representations (Anderson *et al* 1968, Moshinsky and Quesne 1971, Rosensteel and Rowe 1977, Sternberg and Wolf 1978, Kashiwara and Vergne 1978, Rowe *et al* 1985). It is our purpose here to draw

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the attention of theoretical physicists to the salient features of these two overlapping classes of representations and to do so in a manner which involves a natural generalisation to the non-compact groups of a body of techniques widely used in the case of compact groups.

In this respect we allow our prejudices in favour of the use of Schur functions in describing and manipulating characters of finite dimensional unitary irreducible representations of the classical compact Lie groups to be extended to the non-compact case.

Our specific aim is to demonstrate that the characters of certain infinite dimensional unirreps of non-compact Lie groups G may be expressed as an infinite sum of characters of finite dimensional unirreps of the corresponding maximal compact subgroups K of G. Such a relationship gives explicitly the branching rules from G to K, and may be used to determine the irreducible constituents of some tensor products of unirreps of G.

The main prongs of our attack on this problem are based on fundamental results (Harish-Chandra 1956, 1966, Schmid 1975, Hecht and Schmid 1975, Atiyah and Schmid 1977) on the discrete series representations and more recent work on harmonic representations (Kashiwara and Vergne 1978, Rowe *et al* 1985). In this paper we limit ourselves to a study of the non-compact groups $U(p, q)$, $Sp(2n, R)$ and $SO^*(2n, R)$ whose maximal compact subgroups are $U(q)\times U(p)$, $U(n)$ and $U(n)$ respectively. In these three cases the generalised characters of an important class of discrete series representations take a particularly simple form. These representations are referred to as holomorphic discrete series representations.

The organisation of the paper is such that discrete series representations will be identified in § 2 through the specification of their characters on restriction to the maximal compact subgroup. In the case of holomorphic discrete series unirreps of the three families of groups under consideration it is shown in § 3 that these characters may be very succinctly expressed in terms of Schur functions. These are themselves characters of unirreps of the compact unitary groups.

Harmonic series representations of $Sp(2n, R)$ are introduced in § 4, in which the emphasis is placed on the analogy with spin representations of $O(2n)$. The link between harmonic series representations and holomorphic unirreps is made in §§ 5 and 6 through the derivation of $Sp(2n, R)\supset U(n)$ and $U(p, q)\supset U(q)\times U(p)$ branching rules.

The paper closes with two sections concerned with the analysis of tensor products of holomorphic and harmonic series representations. The groups $Sp(2n, R)$ and $U(p, q)$ are discussed in detail. Two rather remarkable conjectures are made, giving closed formulae covering all tensor products considered here. These formulae provide straightforward algorithms involving only Schur function manipulations and modification rules.

2. Root systems and discrete series characters

Let G be a connected reductive non-compact Lie group, K a maximal reductive compact Lie subgroup, and H a Cartan subgroup of G which is contained in K. Corresponding to G, K and H there exist real Lie algebras g, k and h along with their complexifications, g^C, k^C and h^C. The Cartan–Weyl canonical form of g^C with respect to h^C defines a set Σ of roots $\boldsymbol{r}$ of g^C. A root $\boldsymbol{r}$ in Σ is said to be compact if it is a root of k^C and non-compact otherwise. Thus Σ decomposes into the disjoint union of Σ^c and Σ^n, the sets of compact and non-compact roots, respectively.

In what follows a crucial role is played by the Weyl group W of the compact group K. This is the group generated by reflections in the hyperplanes perpendicular to each of the compact roots $\boldsymbol{r}$ in Σ^c.

It is convenient to adopt once and for all some particular ordering of the compact roots $\boldsymbol{r}$ in Σ^c which serves to define the subset Π^c of positive roots. In contrast to this a variety of orderings of the non-compact roots $\boldsymbol{r}$ in Σ^n are required in different circumstances. In each case the subset of positive roots is denoted by Π^n. Further we introduce

$$\boldsymbol{\rho}^c = \tfrac{1}{2} \sum_{\boldsymbol{r}\in\Pi^c} \boldsymbol{r}, \qquad \boldsymbol{\rho}^n = \tfrac{1}{2} \sum_{\boldsymbol{r}\in\Pi^n} \boldsymbol{r} \tag{2.1}$$

and

$$\boldsymbol{\rho} = \boldsymbol{\rho}^c + \boldsymbol{\rho}^n.$$

A vector $\boldsymbol{\lambda}$ is said to be K dominant if $(\boldsymbol{r}, \boldsymbol{\lambda}) > 0$ for all $\boldsymbol{r}$ in the set of positive roots Π^c of K. Equivalently $\boldsymbol{\lambda} \geqslant S\boldsymbol{\lambda}$ for all elements S of the Weyl group W of K.

There exist two lattices of interest: one Λ_0 which can be written as integral linear combinations of the roots $\boldsymbol{r}$ in Σ, and a second Λ displaced from the first by $\boldsymbol{\rho}$. It should be stressed that Λ is independent of the particular ordering adopted in distinguishing between positive and negative roots.

A vector $\boldsymbol{\lambda}$ in Λ is said to be non-singular if $(\boldsymbol{r}, \boldsymbol{\lambda}) \neq 0$ for all $\boldsymbol{r}$ in Σ.

With these conventions there exists for each non-singular $\boldsymbol{\lambda}$ in Λ a unique tempered invariant eigendistribution $\theta_{\boldsymbol{\lambda}}$ which serves as a generalised character of a discrete series unirrep of G (Harish-Chandra 1966, Schmid 1975, Atiyah and Schmid 1977). On restriction to the compact subgroup K this character takes the form

$$\theta_{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \left((-1)^q \sum_{S\in W} \eta_S \exp \mathrm{i}S\boldsymbol{\lambda}\cdot\boldsymbol{\phi}\right)\left(\prod_{\substack{\boldsymbol{r}\in\Sigma \\ (\boldsymbol{r},\boldsymbol{\lambda})>0}} [\exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi}) - \exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi})]\right)^{-1} \tag{2.2}$$

where $\boldsymbol{\phi}$ is an appropriate sequence $(\phi_1, \phi_2, \ldots)$ of real class parameters of K and $q = \frac{1}{2}(\dim \mathrm{g}^{\mathrm{C}} - \dim \mathrm{k}^{\mathrm{C}})$ is the number of non-compact roots for which $(\boldsymbol{r}, \boldsymbol{\lambda}) > 0$. The summation is carried out over all elements S of the Weyl group W of K and η_S is the parity (± 1) of S.

Every $\theta_{\boldsymbol{\lambda}}$ arises as the character of a discrete series unirrep provided that $\boldsymbol{\lambda}$ is non-singular, whilst conversely every discrete series character occurs amongst the set of such $\theta_{\boldsymbol{\lambda}}$ (Harish-Chandra 1966, Schmid 1975, Atiyah and Schmid 1977). Hence all the discrete series unirreps may be specified, up to equivalence, by means of those $\boldsymbol{\lambda}$ in Λ which are non-singular and K-dominant. The latter restriction is convenient since vectors $\boldsymbol{\lambda}$ related by the action of Weyl group elements yield the same character (2.2).

Hence

$$\theta_{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \left(\sum_{S\in W} \eta_S \exp \mathrm{i}S\boldsymbol{\lambda}\cdot\boldsymbol{\phi}\right)\left(\prod_{\boldsymbol{r}\in\Pi^c} [\exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi}) - \exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi})] \right.$$
$$\left. \times \prod_{\substack{\boldsymbol{r}\in\Pi^n \\ (\boldsymbol{r}\cdot\boldsymbol{\lambda})>0}} [\exp(-\tfrac{1}{2}\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi}) - \exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi})]\right)^{-1}. \tag{2.3}$$

For the K dominant vector $\boldsymbol{\lambda}$ the ordering of the non-compact roots is such that the set of positive roots is *defined by*

$$\Pi^n = (\boldsymbol{r} \in \Sigma^n \colon (\boldsymbol{r}, \boldsymbol{\lambda}) > 0\}. \tag{2.4}$$

It then follows from Weyl's character formula for unirreps of the compact Lie group K that

$$\theta_{\lambda}(\phi) = \chi_{\lambda-\rho^c}(\phi) \exp(\mathrm{i}\rho^{\mathrm{n}} \cdot \phi) \prod_{r \in \Pi^{\mathrm{n}}} [1-\exp(\mathrm{i}r \cdot \phi)]^{-1} \tag{2.5}$$

where $\chi_{\lambda-\rho^c}(\phi)$ is the character of the unirrep of K with highest weight vector $\lambda - \rho^c$. The fact that λ is non-singular and K dominant ensures that $\lambda - \rho^c$ is a well defined highest weight vector.

It has been shown (Schmid 1975, Hecht and Schmid 1975) that for all non-singular λ, $\theta_{\lambda}(\phi)$ may be expressed in the form

$$\theta_{\lambda}(\phi) = \sum_{\mu} B_{\lambda}^{\mu} \chi_{\mu}(\phi) \tag{2.6}$$

where the branching rule coefficients B_{λ}^{μ} for the restriction from non-compact G to compact K may be determined through the use of Blattner's conjecture. It is our intention to show that for certain λ the complexities of the corresponding formula (Hecht and Schmid 1975)

$$B_{\lambda}^{\mu} = \sum_{S \in \mathrm{W}} \eta_S Q(S(\mu + \rho^c) - \lambda - \rho^{\mathrm{n}}) \tag{2.7}$$

may be avoided. In this formula as in (2.6), μ is K dominant whilst $Q(\nu)$ is the number of distinct ways in which ν can be expressed as a sum of positive non-compact roots (2.4). It should be pointed out that as a consequence of (2.7) the leading term in (2.6) corresponds to $\mu = \lambda - \rho^c + \rho^{\mathrm{n}}$ and has multiplicity one, whilst the remaining terms involve $\mu = \lambda - \rho^c + \rho^{\mathrm{n}} - \nu$ where ν is a positive combination of non-compact roots.

Thus the discrete series unirrep λ of G could equally well have been labelled by $\mu = \lambda - \rho^c + \rho^{\mathrm{n}}$ which is the highest weight of the leading unirrep of K contained in the restriction of λ from G to K. In this sense μ is the analogue for G of a highest weight label.

3. Holomorphic discrete series representations and Schur functions

The difficulty in exploiting (2.5) directly lies in the facts that ρ^{n} depends upon the ordering induced by λ through the definition (2.4), and that in general ρ^{n} is not invariant under the action of elements S of the Weyl group W.

In certain cases these difficulties may be overcome rather neatly. To illustrate this, consider the non-compact groups $\mathrm{U}(p, q)$, $\mathrm{Sp}(2n, R)$ and $\mathrm{SO}^*(2n)$ whose maximal compact subgroups are $\mathrm{U}(q) \times \mathrm{U}(p)$, $\mathrm{U}(n)$ and $\mathrm{U}(n)$ respectively. Since the Weyl group of $\mathrm{U}(n)$ is the symmetric group, S_n, of all permutations of components of vectors in the root space, the appropriate Weyl groups are $\mathrm{S}_q \times \mathrm{S}_p$, S_n and S_n respectively. The corresponding root systems are specified in table 1 in terms of mutually orthogonal unit vectors e_i and d_a for appropriate values of i and a.

The usual lexicographic ordering of vectors in the root space of the maximal compact subgroup K then fixes the positive compact roots Π^c as given in table 1. Correspondingly

$$\mathrm{U}(p, q) \qquad \rho^c = \left(\frac{p-1}{2}, \frac{p-3}{2}, \ldots, \frac{-p+1}{2}, \frac{q-1}{2}, \frac{q-3}{2}, \ldots, \frac{-q+1}{2}\right) \tag{3.1a}$$

Table 1. Root systems

G	K	Σ^c (compact)		Σ^n (non-compact)		Π^c (positive compact)	
$U(p,q)$	$U(q)\times U(p)$	$\pm(e_i-e_j)$ $\pm(d_a-d_b)$	$1\leq i<j\leq p$ $1\leq a<b\leq q$	$\pm(e_i-d_a)$	$\begin{cases}1\leq i\leq p\\ 1\leq a\leq q\end{cases}$	e_i-e_j d_a-d_b	$1\leq i<j\leq p$ $1\leq a<b\leq q$
$Sp(2n,R)$	$U(n)$	$\pm(e_i-e_j)$	$1\leq i<j\leq n$	$\pm(e_i+e_j)$	$1\leq i\leq j\leq n$	e_i-e_j	$1\leq i<j\leq n$
$SO^*(2n)$	$U(n)$	$\pm(e_i-e_j)$	$1\leq i<j\leq n$	$\pm(e_i+e_j)$	$1\leq i<j\leq n$	e_i-e_j	$1\leq i<j\leq n$

$$\mathrm{Sp}(2n,R)\qquad \boldsymbol{\rho}^c=\left(\frac{n-1}{2},\frac{n-3}{2},\ldots,\frac{-n+1}{2}\right) \tag{3.1b}$$

$$\mathrm{SO}^*(2n)\qquad \boldsymbol{\rho}^c=\left(\frac{n-1}{2},\frac{n-3}{2},\ldots,\frac{-n+1}{2}\right). \tag{3.1c}$$

In contrast $\boldsymbol{\rho}^n$ depends upon $\boldsymbol{\lambda}$ through (2.4). In the case of $U(p,q)$ for example there are precisely $(p+q)!/p!q!$ distinct types (Graev 1968) of non-singular K dominant vectors $\boldsymbol{\lambda}$ each specified by a permutation Π for which

$$\lambda_{\Pi_1}>\lambda_{\Pi_2}>\ldots>\lambda_{\Pi_{p+q}},$$

subject to the constraints

$$\lambda_1>\lambda_2>\ldots>\lambda_p \qquad \text{and} \qquad \lambda_{p+1}>\lambda_{p+2}>\ldots>\lambda_{p+q}$$

corresponding to the K dominant condition. Each such type specifies a type of discrete series unirrep.

It is convenient to concentrate attention on holomorphic discrete series unirreps (Harish-Chandra 1956). A discrete series unirrep $\boldsymbol{\lambda}$ is said to be holomorphic (or antiholomorphic) if and only if (Schmid 1975) for each pair of non-compact roots, $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}_\beta$, in Π^n $\boldsymbol{r}_\alpha+\boldsymbol{r}_\beta$ is not a positive compact root i.e.

$$\boldsymbol{r}_\alpha+\boldsymbol{r}_\beta\notin\Pi^c.$$

The dependence upon $\boldsymbol{\lambda}$ is implicit in the use of (2.4) to define Π^n.

By considering the identities

$$\begin{aligned} e_i-e_j&=(e_i-d_a)+(d_a-e_j) &&\text{for } i<j\\ d_a-d_b&=(d_a-e_i)+(e_i-d_b) &&\text{for } a<b\\ e_i-e_j&=(e_i+e_k)+(-e_k-e_j) &&\text{for } i<j \end{aligned}$$

it is then easy to see that the only possible sets of positive non-compact roots are given by

$$U(p,q)\qquad \Pi^n=\{e_i-d_a\colon 1\leq i\leq p,\ 1\leq a\leq q\} \tag{3.2a}$$

$$\mathrm{Sp}(2n,R)\qquad \Pi^n=\{e_i+e_j\colon 1\leq i\leq j\leq n\} \tag{3.2b}$$

$$\mathrm{SO}^*(2n)\qquad \Pi^n=\{e_i+e_j\colon 1\leq i<j\leq n\} \tag{3.2c}$$

and their complementary sets in Σ^n, namely $\Pi^{n'}=\Sigma^n\backslash\Pi^n$ with Π^n specified as above.

The restrictions on the components of $\boldsymbol{\lambda}$ then take the form

$$U(p,q)\qquad \lambda_1>\lambda_2>\ldots>\lambda_p>\lambda_{p+1}>\ldots>\lambda_{p+q} \tag{3.3a}$$

$$\mathrm{Sp}(2n, R) \qquad \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0 \tag{3.3b}$$

$$\mathrm{SO}^*(2n) \qquad \lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > |\lambda_n| \geqslant 0 \tag{3.3c}$$

for Π^n as given in (3.2) whilst for the complementary cases

$$\mathrm{U}(p, q) \qquad \lambda_{p+1} > \lambda_{p+2} > \ldots > \lambda_{p+q} > \lambda_1 > \lambda_2 \ldots > \lambda_p \tag{3.3a'}$$

$$\mathrm{Sp}(2n, R) \qquad 0 > \lambda_1 > \lambda_2 > \ldots > \lambda_n \tag{3.3b'}$$

$$\mathrm{SO}^*(2n) \qquad 0 \geqslant -|\lambda_1| > \lambda_2 > \ldots > \lambda_n. \tag{3.3c'}$$

These cases correspond to holomorphic discrete series unirreps which are contragredient to those specified by (3.3). The terminology in the literature is somewhat confused but the unirreps of (3.3) and (3.3′) are said to constitute the positive and negative discrete series, D_+ and D_- respectively. The unirreps (3.3) of D_+ have a lowest weight with respect to the ordering of weights appropriate to the maximal compact subgroup, whilst the unirreps (3.3′) of D_- have correspondingly a highest weight. Both sets of unirreps are holomorphic but it is customary (Sternberg and Wolf 1978, Repka 1979) to refer to those of D_+ as antiholomorphic and those of D_- as holomorphic. The convention will *not* be adhered to here. Attention will be restricted from now on to the discrete series unirreps specified by (3.3) which will be referred to as *holomorphic discrete series* unirreps.

For these representations the non-compact analogues of (3.1) then become

$$\mathrm{U}(p, q) \qquad \boldsymbol{\rho}^{\mathrm{n}} = \left(\frac{q}{2}, \frac{q}{2}, \ldots, \frac{q}{2}, -\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2}\right) \tag{3.4a}$$

$$\mathrm{Sp}(2^n, R) \qquad \boldsymbol{\rho}^{\mathrm{n}} = \left(\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2}\right) \tag{3.4b}$$

$$\mathrm{SO}^*(2n) \qquad \boldsymbol{\rho}^{\mathrm{n}} = \left(\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2}\right). \tag{3.4c}$$

These vectors are remarkable for the fact that they are invariant under the action of the elements S of the Weyl group W allowing (2.5) to be simplified to

$$\theta_{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \chi_{\boldsymbol{\lambda}-\boldsymbol{\rho}^{\mathrm{c}}+\boldsymbol{\rho}^{\mathrm{n}}}(\boldsymbol{\phi}) \prod_{\boldsymbol{r}\in\Pi^{\mathrm{n}}} [1-\exp(\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{\phi})]^{-1}. \tag{3.5}$$

Moreover the final term in this expression takes the form

$$\mathrm{U}(p, q) \qquad \prod_{1\leqslant j\leqslant p}\prod_{1\leqslant a\leqslant q} (1-\mathrm{e}^{\mathrm{i}\phi_j}\,\mathrm{e}^{-\mathrm{i}\phi_{p+a}})^{-1} \tag{3.6a}$$

$$\mathrm{Sp}(2n, R) \qquad \prod_{1\leqslant i\leqslant j\leqslant n} (1-\mathrm{e}^{\mathrm{i}\phi_i}\,\mathrm{e}^{\mathrm{i}\phi_j})^{-1} \tag{3.6b}$$

$$\mathrm{SO}^*(2n) \qquad \prod_{1\leqslant i< j\leqslant n} (1-\mathrm{e}^{\mathrm{i}\phi_i}\,\mathrm{e}^{\mathrm{i}\phi_j})^{-1}. \tag{3.6c}$$

These terms (3.6) are nothing other than rather well known generating functions for specific infinite series of characters of unitary groups as given by Littlewood (1940, p 238). The only slight complication being the presence of a negative sign in the exponent in (3.6*a*): this invokes complex conjugate characters.

Thus (3.6*a*) is just the generating function for the infinite series of Schur functions (*S* functions)

$$\sum_{\zeta} \{\zeta\} \cdot (\bar{\zeta}\} \tag{3.7a}$$

where the summation is over all partitions (ζ), (3.6*b*) that for the infinite *S* function series

$$D = \sum_{\delta} \{\delta\} \tag{3.7b}$$

where the summation is over all partitions (δ) whose parts are even, and (3.6*c*) that for the infinite *S* function series

$$B = \sum_{\beta} \{\beta\} \tag{3.7c}$$

where the summation is over all partitions (β) whose parts are repeated an even number of times. The notation and properties of the various infinite series of *S* functions have been explained fully elsewhere (King 1975, King *et al* 1981, Black *et al* 1983).

Writing

$$\mathrm{U}(p, q) \qquad \boldsymbol{\lambda} - \boldsymbol{\rho}^{\mathrm{c}} + \boldsymbol{\rho}^{\mathrm{n}} = \kappa - \nu = (\kappa_1, \kappa_2, \ldots, \kappa_p, -\nu_q, \ldots, -\nu_2, -\nu_1) \tag{3.8a}$$

$$\mathrm{Sp}(2n, R) \qquad \boldsymbol{\lambda} - \boldsymbol{\rho}^{\mathrm{c}} + \boldsymbol{\rho}^{\mathrm{n}} = \mu = (\mu_1, \mu_2, \ldots, \mu_n) \tag{3.8b}$$

$$\mathrm{SO}^*(2n) \qquad \boldsymbol{\lambda} - \boldsymbol{\rho}^{\mathrm{c}} + \boldsymbol{\rho}^{\mathrm{n}} = \mu = (\mu_1, \mu_2, \ldots, \mu_n), \tag{3.8c}$$

the holomorphic discrete series unirreps may be labelled by using the partitions κ, ν and μ as follows

$$\mathrm{U}(p, q) \qquad \{\{\bar{\nu}; \kappa\}\} \qquad \text{with } \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_p, \ \nu_1 \geq \nu_2 \geq \ldots \geq \nu_q \text{ and}$$

$$\kappa_p + \nu_q \geq p + q \tag{3.9a}$$

$$\mathrm{Sp}(2n, R) \qquad \langle\{\mu\}\rangle \qquad \text{with } \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > n \tag{3.9b}$$

$$\mathrm{SO}^*(2n) \qquad [\{\mu\}] \qquad \text{with } \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq |\mu_n - n + 1| + (n-1), \tag{3.9c}$$

where the double brackets have been used to emphasise that these representations of non-compact groups G are being labelled by the highest weights of representations of the corresponding maximal compact subgroups K.

The corresponding characters take the form

$$\{\{\bar{\nu}; \kappa\}\} \qquad \theta_{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \sum_{\zeta \in F} \chi_{\kappa}(\theta) \chi_{\bar{\nu}}(\psi) \chi_{\zeta}(\theta) \chi_{\bar{\zeta}}(\psi) \tag{3.10a}$$

with $\boldsymbol{\phi} = (\boldsymbol{\theta}, \boldsymbol{\psi})$

$$\langle\{\mu\}\rangle \qquad \theta_{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \sum_{\delta \in D} \chi_{\mu}(\boldsymbol{\phi}) \chi_{\delta}(\boldsymbol{\phi}) \tag{3.10b}$$

$$[\{\mu\}] \qquad \theta_{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \sum_{\beta \in B} \chi_{\mu}(\boldsymbol{\phi}) \chi_{\beta}(\boldsymbol{\phi}) \tag{3.10c}$$

where in (3.10*a*) the infinite *S* function series

$$F = \sum_{\zeta} \{\zeta\} \tag{3.11}$$

involves a summation over all partitions ζ.

It should further be stressed that these characters automatically give the branching rules appropriate for the restriction from the non-compact group G to the maximal compact subgroup K. Taking note of the fact that $\chi_\mu(\phi)$ is the character of the unirrep $\{\mu\}$ of the unitary group $U(n)$, and similarly for the other factors appearing in (3.10), these branching rules can be seen to take the form

$$U(p, q) \to U(q) \times U(p) \qquad \{\{\bar{\nu}; \kappa\}\} \to \sum_{\zeta \in F} \{\bar{\nu}\} \cdot \{\bar{\zeta}\} \times \{\kappa\} \cdot \{\zeta\} \tag{3.12a}$$

$$Sp(2n, R) \to U(n) \qquad \langle\{\mu\}\rangle \to \sum_{\delta \in D} \{\mu\} \cdot \{\delta\} = \{\mu\} \cdot D \tag{3.12b}$$

$$SO^*(2n) \to U(n) \qquad [\{\mu\}] \to \sum_{\beta \in B} \{\mu\} \cdot \{\beta\} = \{\mu\} \cdot B. \tag{3.12c}$$

The result (3.12*a*) has been given implicitly in a determinantal form by Graev (1968). The other expressions are, however, new and all three are surprisingly simple.

4. Harmonic series representations

Just as there exists a spin representation Δ of the orthogonal group $SO(2n)$ associated with a Clifford algebra, so there also exists a harmonic representation, $\tilde{\Delta}$, of the non-compact symplectic group $Sp(2n, R)$ associated with a Heisenberg algebra. Moreover Δ is a true, unitary, finite dimensional representation of the double covering group $Spin(2n)$ of $SO(2n)$, whilst $\tilde{\Delta}$ is a true, unitary, infinite dimensional representation of the double covering group $Mp(2n)$ of $Sp(2n, R)$, the so-called metaplectic group.

The representation Δ is reducible into the sum of two irreducible representations, Δ_+ and Δ_-, of $SO(2n)$ whose highest weights are $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2}\frac{1}{2})$ and $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2}-\frac{1}{2})$, corresponding to highest weights of the representations $\varepsilon^{1/2}\{0\}$ and $\varepsilon^{1/2}\{\bar{1}\}$ of $U(n)$ which appear in their restriction to this maximal subgroup.

In a very similar way the representation $\tilde{\Delta}$ is reducible into the sum of two irreducible representations $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ of $Sp(2n, R)$ whose leading weights are $(\frac{1}{2}\frac{1}{2}\ldots\frac{1}{2})$ and $(\frac{3}{2}\frac{1}{2}\ldots\frac{1}{2})$ (Moshinsky and Quesne 1971), corresponding to the highest weights of the representations $\varepsilon^{1/2}\{0\}$ and $\varepsilon^{1/2}\{1\}$ of $U(n)$ which appear in their restriction to this maximal compact subgroup.

Continuing the analogy, the tensor powers Δ^k decompose into direct sums of unitary irreducible representations of $SO(2n)$ (or more properly of $Spin(2n)$ if k is odd). Indeed all such finite dimensional unirreps appear as a constituent of Δ^k for some k. Similarly the tensor powers $\tilde{\Delta}^k$ all decompose into a direct sum of unirreps of $Sp(2n, R)$ (or more properly of $Mp(2n)$ if k is odd). We shall refer to all those infinite dimensional unirreps which appear as a constituent of $\tilde{\Delta}^k$ for some k, as *harmonic series representations.*

In order to keep track of the origin of a harmonic series representation it has been found useful to label $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ by the symbols $\langle\frac{1}{2}(0)\rangle$ and $\langle\frac{1}{2}(1)\rangle$ respectively, whilst all those unirreps appearing in $\tilde{\Delta}^k$ are labelled by the symbols $\langle\frac{1}{2}k(\lambda)\rangle$.

The justification for this notation lies in the fact that under the restriction from $Sp(2n, R)$ to $U(n)$

$$\tilde{\Delta}_+ = \langle\tfrac{1}{2}(0)\rangle \to \varepsilon^{1/2}(\{0\} + \{2\} + \{4\} + \ldots) \tag{4.1a}$$

$$\tilde{\Delta}_- = \langle\tfrac{1}{2}(1)\rangle \to \varepsilon^{1/2}(\{1\} + \{3\} + \{5\} + \ldots) \tag{4.1b}$$

and more generally

$$\langle\tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{k/2}\left(\sum_{\mu} R^{\mu}_{\lambda}\{\mu\}\right) = \varepsilon^{k/2}(\{\lambda\}+\ldots) \tag{4.2}$$

where the terms indicated by ... all have higher highest weights than the representation $\{\lambda\}$. Moreover the harmonic series representations appearing in $\tilde{\Delta}^k$ are in one-to-one correspondence with the terms arising in the branching rule appropriate to the restriction from Sp(2nk, R) to Sp(2n, R) $\times$ O(k)

$$\tilde{\Delta} \to \sum_{\zeta} \langle\tfrac{1}{2}k(\lambda)\rangle \times [\lambda] \tag{4.3}$$

where the summation is carried out over all those partitions $(\lambda)=(\lambda_1, \lambda_2, \ldots)$ for which the conjugate partition $(\tilde{\lambda})=(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ satisfies the constraints

$$\tilde{\lambda}_1+\tilde{\lambda}_2 \leqslant k \tag{4.4a}$$

and

$$\tilde{\lambda}_1 \leqslant n. \tag{4.4b}$$

This result (4.3) was first given by Kashiwara and Vergne (1978) and the constraints (4.4), which were proved to cover all unirreps of Sp(2n, R) having a lowest U(n) weight by Rowe *et al* (1985), imply that the summation is over precisely those representation labels of O(k) which are standard (Hamermesh 1962, p 394) and which also label irreducible covariant tensor representations of U(n).

For the interested reader it might be pointed out that (4.1a) and (4.1b) are analogues of the SO(2n) to U(n) branching rules

$$\Delta_+ \to \begin{cases} \varepsilon^{1/2}(\{0\}+\{1^2\}+\ldots+\{1^n\}) & n \text{ even} \\ \varepsilon^{-1/2}(\{1\}+\{1^3\}+\ldots+\{1^n\}) & n \text{ odd} \end{cases} \tag{4.5}$$

$$\Delta_- \to \begin{cases} \varepsilon^{-1/2}(\{1\}+\{1^3\}+\ldots+\{1^{n-1}\}) & n \text{ even} \\ \varepsilon^{-1/2}(\{0\}+\{1^2\}+\ldots+\{1^{n-1}\}) & n \text{ odd} \end{cases} \tag{4.6}$$

whilst the analogue of (4.3) is the branching rule for the restriction from SO(2nk) to SO(2n) $\times$ SO(k) which can be written in the form

$$\Delta \to \sum_{\lambda} [\tfrac{1}{2}k(\lambda)] \times [\lambda] \tag{4.7}$$

with the representation of SO(2n) signified here by $[\tfrac{1}{2}k(\lambda)]$ recast in the standard labelling format through the identification

$$[\tfrac{1}{2}k(\lambda)] = \begin{cases} [m^n/\tilde{\lambda}] & \text{if } k=2m \\ [\Delta;\, m^n/\tilde{\lambda}] & \text{if } k=2m+1. \end{cases} \tag{4.8}$$

The summation in (4.7) is carried out over all those partitions (λ) for which the conjugate partition $(\tilde{\lambda})$ satisfies the constraints

$$\tilde{\lambda}_1 \leqslant k \tag{4.9a}$$

and

$$\lambda_1 \leqslant n. \tag{4.9b}$$

This rule (4.7), established first by Morris (1958), involves certain representations of SO($2n$) and SO($2k$) which are reducible namely those of SO($2n$) arising when $\lambda_1 = n$ and those of SO($2k$) arising when $\lambda_1 < n$ and those of SO($2k$) arising when $\tilde{\lambda}_1 = k$.

One important distinction between the non-compact and compact cases lies in the fact that (4.1) to (4.3) involve infinite series whilst (4.5) to (4.7) give finite series.

5. The Sp(2*n*, *R*) → U(*n*) branching rule

In order to understand the true nature of the unirreps $\langle \frac{1}{2}k(\lambda)\rangle$ of Sp($2n, R$) it is necessary to determine explicitly the branching rule (4.3) appropriate to the restriction from Sp($2n, R$) to U(n) and thereby evaluate the corresponding group character on the conjugacy classes of U(n). This has been carried out (Rowe *et al* 1985) through a consideration of the group-subgroup chains

$$\mathrm{Sp}(2nk, R) \supset \mathrm{Sp}(2n, R) \times \mathrm{O}(k) \supset \mathrm{U}(n) \times \mathrm{O}(k) \tag{5.1}$$

and

$$\mathrm{Sp}(2nk, R) \supset \mathrm{U}(nk) \supset \mathrm{U}(n) \times \mathrm{U}(k) \supset \mathrm{U}(n) \times \mathrm{O}(k). \tag{5.2}$$

Consideration of the first gives, from (4.3) and (4.2),

$$\tilde{\Delta} \to \sum_{\lambda} \langle \tfrac{1}{2}k(\lambda)\rangle \times [\lambda] \to \sum_{\lambda}\sum_{\mu} \varepsilon^{k/2} R^{\mu}_{\lambda}\{\mu\} \times [\lambda] \tag{5.3}$$

whilst the second yields

$$\tilde{\Delta} \to \sum_{m} \varepsilon^{1/2} \cdot [1^m\} \to \sum_{\rho} \varepsilon^{k/2} \cdot \{\rho\} \times \varepsilon^{n/2} \cdot \{\rho\}$$

$$\to \sum_{\rho} \varepsilon^{k/2} \cdot \{\rho\} \times [\rho/D] = \sum_{\rho} \varepsilon^{k/2}\{\zeta \cdot D\} \times [\zeta]. \tag{5.4}$$

In (5.3) the summation involves a single term associated with each linearly independent character $[\lambda]$ of O(k) specified unambiguously by a standard label (λ). Unfortunately the manipulations of (5.4) involve, in general, non-standard labels $[\zeta]$ for which modification rules (King 1971, 1975, Black *et al* 1983) must be applied. The net result for the restriction Sp($2n, R$) → U(n) takes the form

$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{k/2} \cdot \{\lambda_s\}^k \cdot D \tag{5.5}$$

where $\{\lambda_s\}^k$ is a *signed sequence* (Rowe *et al* 1985) of terms $\pm\{\rho\}$ such that $\pm[\rho]$ is equivalent to $[\lambda]$ under the modification rules of O(k).

For example

$$\{54_s\}^4 = \{54\} - \{542\} + \{5431\} - \{543^2\} - \{54^21^2\} + \ldots.$$

It is to be noted however that in (5.5) the products signified by · are tensor products to be carried out in the group U(n). This imposes its own limits on the terms appearing in each of the infinite sequences $\{\lambda_s\}^k$ and D. In fact the former is rendered finite although the latter remains infinite. For example

$$\{54_s\}^4_3 = \{54\} - \{542\}$$

where the subscript n on $\{\lambda_s\}^k_n$ indicates that only partitions with n or fewer parts are to be retained. Similarly we write D_n to signify a similar constraint on D. For example

$$D_2 = \{0\} + \{2\} + \{4\} + \{2^2\} + \{6\} + \{42\} + \{8\} + \{62\} + \{4^2\} + \{10\} + \ldots.$$

Of course as is implicit in (5.4) the value of k also places limits on the number of parts of the partitions ρ and ζ and on the relevant terms of D. Thus (5.5) can be replaced by

$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{k/2} \cdot \{\{\lambda_s\}_N^k \cdot D_N\}_N \tag{5.6}$$

with $N = \min(n, k)$. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$ as implied by the final subscript N. This result has been derived and exemplified elsewhere (Rowe *et al* 1985).

The modification rules of $O(k)$ (King 1971, 1975, Black *et al* 1983) imply that the signed sequence

$$\{\lambda_s\}^k = \{\lambda\} - \{\mu\} + \{\nu\} \pm \{\rho\} + \dots \tag{5.7}$$

is such that

$$\tilde{\lambda}_1 < \tilde{\mu}_1 < \tilde{\nu}_1 \leqslant \tilde{\rho}_1 \dots .$$

Moreover

$$\{\tilde{\mu}\} = \begin{cases} \{k+1-\tilde{\lambda}_2, \tilde{\lambda}_1+1, \tilde{\lambda}_3, \tilde{\lambda}_4, \dots\} & \text{if } \tilde{\lambda}_1 \leqslant [k/2] \\ \{k+1-\tilde{\lambda}_2, k+1-\tilde{\lambda}_1, \tilde{\lambda}_3, \tilde{\lambda}_4, \dots\} & \text{if } \tilde{\lambda}_1 > [k/2] \end{cases} \tag{5.8}$$

so that $\tilde{\mu}_1 = k+1-\tilde{\lambda}_2$. It follows that the branching formula (5.6) for the restriction from $Sp(2n, R)$ to $U(n)$ takes the very simple form

$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{k/2} \cdot \{\lambda\} \cdot D \tag{5.9}$$

if and only if

$$\tilde{\lambda}_1 \leqslant n \leqslant k - \tilde{\lambda}_2 \tag{5.10}$$

where it should be stressed that the products on the right-hand side of (5.9) are now to be evaluated in $U(n)$, since $N = \min(n, k) = n$.

Just as the harmonic series unirrep $\langle \frac{1}{2}k(\lambda)\rangle$ of $Sp(2n, R)$ is said to be standard if (4.4*a*) and (4.4*b*) are satisfied, it is convenient to say that it is *highly standard* if and only if the stronger condition (5.10) is satisfied. Thus for example $\langle 2(32)\rangle$ is standard but not highly standard for $Sp(6, R)$. Correspondingly

$$\{32_s\}_3^4 = \{32\} - \{32^2\}$$

and the restriction from $Sp(6, R)$ to $U(3)$ takes the form

$$\begin{aligned} \langle 2(32)\rangle &\to \varepsilon^2 \cdot (\{32\} - \{32^2\}) \cdot D \\ &= \varepsilon^2 \cdot (\{32\} + \{331\} + \{421\} + \{43\} + \{432\} + \{441\} + \dots \\ &= \{542\} + \{553\} + \{643\} + \{652\} + \{654\} + \{663\} + \dots . \end{aligned} \tag{5.11}$$

In contrast to this $\langle 2(32)\rangle$ is standard and highly standard for $Sp(4, R)$. In this case

$$\{32_s\}_2^4 = \{32\}$$

and the restriction from $Sp(4, R)$ to $U(2)$ yields

$$\begin{aligned} \langle 2(32)\rangle &\to \varepsilon^2 \cdot \{32\} \cdot D \\ &= \varepsilon^2 \cdot (\{32\} + \{43\} + \{52\} + \{54\} + \{63\} + \{65\} + \dots \\ &= \{54\} + \{65\} + \{74\} + \{76\} + \{85\} + \{87\} + \dots . \end{aligned} \tag{5.12}$$

Incidentally, whether or not $\langle\frac{1}{2}k(\lambda)\rangle$ is highly standard for Sp($2n$, R) the coefficients R^{μ}_{λ} appearing in (5.3) can be seen on comparison with (5.4) to be identical with those appearing in the branching rule from U(k) to O(k)

$$\{\mu\}\rightarrow\sum_{\lambda} R^{\mu}_{\lambda}[\lambda]. \tag{5.13}$$

This may be used to check individual coefficients in such results as (5.11) and (5.12). At first sight this looks an extremely helpful result but the problem of modification rules has simply been transferred elsewhere. Now, in evaluating (5.13), modification rules must be used to express the result in terms of O(k) standard labels $[\lambda]$. However the result does not readily give the required expression

$$\langle\tfrac{1}{2}k(\lambda)\rangle\rightarrow\sum_{\mu} \varepsilon^{k/2}R^{\mu}_{\lambda}\{\mu\} \tag{5.14}$$

for the Sp($2n$, R) restriction to U(n) since the summation in (5.13) is over λ rather than μ as in (5.14).

Since unirreps of Sp($2n$, R) yielding the same character on restriction to U(n) are necessarily equivalent, the real importance of (5.9) is that it implies the equivalence of certain harmonic series unirreps $\langle\frac{1}{2}k(\lambda)\rangle$ of Sp($2n$, R) for various k and (λ). These equivalences can be seen by noting that (5.9) gives

$$\langle\tfrac{1}{2}k(\lambda)\rangle\rightarrow\begin{cases}\{\mu\}\cdot D & \text{for } k=2m \qquad (5.15a)\\ \varepsilon^{1/2}\{\mu\}\cdot D & \text{for } k=2m+1 \qquad (5.15b)\end{cases}$$

with

$$(\mu)=(\mu_1,\mu_2,\ldots,\mu_n)=(\lambda_1+m,\lambda_2+m,\ldots,\lambda_n+m). \tag{5.16}$$

Thus for example, in the case of Sp(4, R) we have the equivalent representations

$$\langle 4(10)\rangle=\langle 3(21)\rangle=\langle 2(32)\rangle \tag{5.17}$$

all branching to the U(2) representations

$$\begin{aligned}\varepsilon^4\cdot\{1\}\cdot D&=\{54\}\cdot D\\ &=\{54\}+\{65\}+\{74\}+\{76\}+\{85\}+\{87\}+\ldots.\end{aligned} \tag{5.18}$$

It is to be noted that (5.17) cannot be extended to include $\langle 1(43)\rangle$ since this violates (5.10). Indeed it violates (4.4) and the representation $\langle 1(43)\rangle$ is not standard.

A further example is provided by the equivalence

$$\langle\tfrac{9}{2}(11)\rangle=\langle\tfrac{7}{2}(221)\rangle \tag{5.19}$$

in Sp(6, R), with both representations branching to

$$\begin{aligned}\varepsilon^{9/2}\cdot\{1^2\}\cdot D&=\varepsilon^{1/2}\cdot\{554\}\cdot D\\ &=\varepsilon^{1/2}(\{554\}+\{754\}+\{655\}+\ldots\end{aligned} \tag{5.20}$$

in U(3). Again (5.19) may not be extended to include $\langle\frac{5}{2}(332)\rangle$ which is inadmissible by virtue of (4.4).

The final point to note is that comparison with the discussion of holomorphic discrete series unirreps of Sp($2n$, R) given in § 3 indicates that such unirreps are all equivalent to harmonic series unirreps. This can be seen by comparing (3.12b) with (5.9).

First of all each holomorphic discrete series unirrep $\langle\{\mu\}\rangle$ necessarily has $\mu_n > n$ and therefore coincides with the highly standard harmonic series unirrep $\langle\frac{1}{2}k(\lambda)\rangle$ where

$$\begin{aligned}\langle\{\mu\}\rangle &= \langle\{\mu_1\mu_2\ldots\mu_n\}\rangle \\ &= \langle\mu_n(\mu_1-\mu_n, \mu_2-\mu_n, \ldots, \mu_{n-1}-\mu_n, 0)\rangle \\ &= \langle\tfrac{1}{2}k(\lambda)\rangle \end{aligned} \tag{5.21}$$

so that $k = 2\mu_n$ and $\tilde{\lambda}_1 \leq n-1 < n < \mu_n < 2\mu_n - \mu_n = k - \mu_n < k - n < k - \tilde{\lambda}_2$ in agreement with (5.10).

Conversely if k is even, say $2m$, and $\lambda_n > n - m$ then the harmonic series unirreps $\langle\frac{1}{2}k(\lambda)\rangle$ is equivalent to a holomorphic discrete series unirrep $\langle\{\mu\}\rangle$ through the identification

$$\begin{aligned}\langle\tfrac{1}{2}k(\lambda)\rangle &= \langle m(\lambda)\rangle \\ &= \langle\{\lambda_1+m, \lambda_2+m, \ldots, \lambda_n+m\}\rangle \\ &= \langle\{\mu\}\rangle\end{aligned}$$

with $\mu_n = \lambda_n + m > n$ as required.

The set of harmonic series unirreps $\langle\frac{1}{2}k(\lambda)\rangle$ thus contains the set of holomorphic discrete series unirreps $\langle\{\mu\}\rangle$ along with others, with $\frac{1}{2}k$ half-odd integral, associated with the metaplectic covering group $\mathrm{Mp}(2n, R)$ of $\mathrm{Sp}(2n, R)$ and still others with $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq k < \tilde{\lambda}_2 + n$ and integer k which correspond to limits of discrete series unirreps or mock-discrete unirreps or ladder unirreps.

6. The $\mathrm{U}(p, q) \to \mathrm{U}(q) \times \mathrm{U}(p)$ branching rule

Turning to the non-compact groups $\mathrm{U}(p, q)$ these may be embedded in $\mathrm{Sp}(2p+2q, R)$ whose harmonic representation $\tilde{\Delta}$ decomposes in accordance with the rule

$$\tilde{\Delta} \to H = H_0 + \sum_{m=1}^{\infty} (H_m + H_{-m}) \tag{6.1}$$

where in what follows it is convenient to write

$$H_0 = \{1(\bar{0}; 0)\} \tag{6.2a}$$

$$H_m = \{1(\bar{0}; m)\} \qquad m = 1, 2, \ldots \tag{6.2b}$$

$$H_{-m} = \{1(\bar{m}; 0)\} \qquad m = 1, 2, \ldots. \tag{6.2c}$$

The unirrep H plays for $\mathrm{U}(p, q)$ the same role as $\tilde{\Delta}$ for $\mathrm{Sp}(2n, R)$, whilst the unirreps H_m are analogues of $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$. The branching rule for the restriction of these fundamental harmonic series unirreps of $\mathrm{U}(p, q)$ to the maximal compact subgroup $\mathrm{U}(q) \times \mathrm{U}(p)$ take the form

$$H_0 = \{1(\bar{0}; 0)\} \to (0 \times \varepsilon) \cdot (\{0\} \times \{0\} + \{\bar{1}\} \times \{1\} + \{\bar{2}\} \times \{2\} + \ldots) \tag{6.3a}$$

$$H_m = \{1(\bar{0}; m)\} \to (0 \times \varepsilon) \cdot (\{0\} \times \{m\} + \{\bar{1}\} \times \{m+1\} + \{\bar{2}\} \times \{m+2\} + \ldots) \tag{6.3b}$$

$$H_{-m} = \{1(\bar{m}; 0)\} \to (0 \times \varepsilon) \cdot (\{\bar{m}\} \times \{0\} + \{\overline{m+1}\} \times \{1\} + \{\overline{m+2}\} \times \{2\} + \ldots). \tag{6.3c}$$

Just as for $\mathrm{Sp}(2n, R)$ the harmonic series unirreps $\langle\frac{1}{2}k(\lambda)\rangle$ are generated by considering the powers $\tilde{\Delta}^k$ of $\tilde{\Delta}$, so the harmonic series unirreps $\{k(\bar{\nu}; \mu)\}$ of $\mathrm{U}(p, q)$ are generated by considering the powers H^k of H. The justification for the notation in the case $k=1$ is provided by the above branching rules. More generally on restriction from U(p, q) to U$(q)\times$U(p) we have

$$\{k(\bar{\nu};\mu)\}\to(0\times\varepsilon^k)\cdot\sum_{\sigma,\tau}R^{\bar{\tau};\sigma}_{\bar{\nu};\mu}\{\bar{\tau}\}\times\{\sigma\}=(0\times\varepsilon^k)\cdot(\{\bar{\nu}\}\times\{\mu\}+\ldots). \tag{6.4}$$

Thanks to the work of Kashiwara and Vergne (1978) these harmonic series unirreps may be studied by noting that on the restriction from U(pk, qk) to U$(p, q)\times$U(k)

$$H\to\sum_{\nu,\mu}\{k(\bar{\nu};\mu)\}\times\{\bar{\nu};\mu\} \tag{6.5}$$

where the summation is carried out over all those pairs of partitions $(\mu)=(\mu_1, \mu_2, \ldots, \mu_p)$, $(\nu)=(\nu_1, \nu_2, \ldots, \nu_q)$ for which the conjugate partitions $(\tilde{\mu})$ and $(\tilde{\nu})$ satisfy the constraints

$$\tilde{\mu}_1+\tilde{\nu}_1\leqslant k \tag{6.6a}$$

and

$$\tilde{\mu}_1\leqslant p \qquad \text{and} \qquad \tilde{\nu}_1\leqslant q. \tag{6.6b}$$

This result was established by Kashiwara and Vergne (1978) and the constraints imply that the summation is carried out over just those mixed tensor representation labels of U(k) which are standard (King 1970, 1975) and which also label irreducible contravarient and covariant tensor representations of U(q) and U(p) respectively.

Following the pattern of § 5 a consideration of the chains

$$\mathrm{U}(pk, qk)\supset\mathrm{U}(p,q)\times\mathrm{U}(k)\supset\mathrm{U}(q)\times\mathrm{U}(p)\times\mathrm{U}(k) \tag{6.7}$$

and

$$\begin{aligned}\mathrm{U}(pk, qk)&\supset\mathrm{U}(qk)\times\mathrm{U}(pk)\supset\mathrm{U}(q)\times\mathrm{U}(k)\times\mathrm{U}(p)\times\mathrm{U}(k)\\&\supset\mathrm{U}(q)\times\mathrm{U}(p)\times\mathrm{U}(k)\end{aligned} \tag{6.8}$$

gives for the first

$$H\to\sum_{\nu,\mu}\{k(\bar{\nu};\mu)\}\times\{\bar{\nu};\mu\}\to\sum_{\substack{\nu,\mu\\\sigma,\tau}}(0\times\varepsilon^k\times0)\cdot R^{\bar{\tau};\sigma}_{\bar{\nu};\mu}(\{\bar{\tau}\}\times\{\sigma\}\times\{\bar{\nu};\mu\}) \tag{6.9}$$

whilst the second yields

$$\begin{aligned}H&\to\sum_{m,n}(0\times\varepsilon)\cdot\{\bar{m}\}\times\{n\}\\&\to\sum_{\xi,\eta}(0\times0\times\varepsilon^k\times\varepsilon^q)\cdot(\{\bar{\xi}\}\times\{\bar{\xi}\}\times\{\eta\}\times\{\eta\})\\&\to\sum_{\xi,\eta}(0\times\varepsilon^k\times\varepsilon^q)\cdot(\{\bar{\xi}\}\times\{\eta\}\times(\{\bar{\xi}\}\cdot\{\eta\}))\\&=\sum_{\xi,\eta,\rho}(0\times\varepsilon^k\times\varepsilon^q)\cdot(\{\bar{\xi}\}\times\{\eta\}\times\{\overline{\xi/\rho};\eta/\rho\})\\&=\sum_{\zeta,\eta,\xi}(0\times\varepsilon^k\times\varepsilon^q)\cdot\{\overline{\varepsilon\cdot\zeta}\}\times\{\eta\cdot\zeta\}\times\{\bar{\xi};\eta\}).\end{aligned} \tag{6.10}$$

Unfortunately the techniques used for deriving (6.10) involve the possibility of non-standard labels $\{\bar{\zeta};\eta\}$ of U(k) arising. These have to be modified in order to recast the result in terms of those standard labels $\{\bar{\nu};\mu\}$ of U(k) appearing in (6.9).

As a result the restriction from U(p, q) to U(q)×U(p) yields the branching rule

$$\{k(\bar{\nu};\mu)\}\to\sum_{\zeta}(0\times\varepsilon^{k})\cdot\{\bar{\nu}_{s};\mu_{s}\}^{k}\cdot\{\bar{\zeta}\}\times\{\zeta\} \tag{6.11}$$

where $\{\bar{\nu}_s;\mu_s\}^k$ is a signed sequence of terms $\pm\{\bar{\sigma}\}\times\{\tau\}$ such that $\pm\{\bar{\sigma};\tau\}$ is equivalent to $\{\bar{\nu};\mu\}$ under the modification rules of U(k).

For example

$$\{\bar{3}_s;1_s\}^2=\{\bar{3}\}\times\{1\}-\{\overline{31}\}\times\{1^2\}+\{\overline{32}\}\times\{1^3\}+\dots.$$

Again this may be rendered finite in (6.11) through the imposition of constraints imposed by specific values of p and q. For example

$$\{\bar{3}_s;1_s\}^2_{2,2}=\{\bar{3}\}\times\{1\}-\{\overline{31}\}\times\{1^2\}$$

where the subscripts q and p on $(\bar{\nu}_s;\mu_s\}^k_{q,p}$ limit the number of parts of the partitions σ and τ, respectively, labelling terms $\pm\{\bar{\sigma}\}\times\{\tau\}$ in the signed sequence. The number of parts of the partition (ζ) appearing in (6.11) are also constrained by both q and p, as are the terms in the tensor products evaluated for U(q) and U(p).

The generalisation of (5.6) appropriate to the restriction from U(p, q) to U(q)× U(p) then takes the form

$$\{k(\bar{\nu};\mu)\}\to\sum_{\zeta}(0\times\varepsilon^{k})\cdot\{\{\bar{\nu}_{s};\mu_{s}\}^{k}_{Q,P}\cdot(\{\bar{\zeta}\}_{Q}\times\{\zeta\}_{P})\}_{Q,P} \tag{6.12}$$

with $P=\min(p,k)$ and $Q=\min(q,k)$.

The modification rules under U(k) (King 1975, Black *et al* 1983) imply that

$$\{\bar{\nu}_s;\mu_s\}^k=\{\bar{\nu}\}\times\{\mu\}-\{\bar{\tau}\}\times\{\sigma\}\pm\{\bar{\kappa}\}\times\{\lambda\}\dots \tag{6.13}$$

with

$$\tilde{\mu}_1<\tilde{\sigma}_1<\tilde{\lambda}_1\leqslant\dots$$
$$\tilde{\nu}_1<\tilde{\tau}_1\leqslant\tilde{\kappa}_1\leqslant\dots.$$

Moreover

$$\{\tilde{\tau}\}=\{k+1-\tilde{\mu}_1,\tilde{\nu}_2,\tilde{\nu}_3,\dots\}\qquad\text{and}\qquad\{\tilde{\sigma}\}=\{k+1-\tilde{\nu}_1,\tilde{\mu}_2,\tilde{\mu}_3,\dots\}. \tag{6.14}$$

It then follows that the branching rule (6.12) reduces to

$$\begin{aligned}\{k(\bar{\nu};\mu)\}&\to\sum_{\zeta}(0\times\varepsilon^{k})\cdot(\{\bar{\nu}\}\times\{\mu\})\cdot(\{\bar{\zeta}\}\times\{\zeta\})\\&=\sum\{\bar{\nu}\}\cdot\{\bar{\zeta}\}\times\varepsilon^{k}\cdot\{\mu\}\cdot\{\zeta\}\end{aligned} \tag{6.15}$$

if and only if both conditions

$$\tilde{\mu}_1\leqslant p\qquad\text{and}\qquad\tilde{\nu}_1\leqslant q \tag{6.16a}$$

are satisfied, along with at least one of the conditions

$$p\leqslant k-\tilde{\nu}_1\qquad\text{and}\qquad q\leqslant k-\tilde{\mu}_1 \tag{6.16b}$$

where the products in (6.15) to the left and right of the × are tensor products to be evaluated in U(q) and U(p) respectively, since $P=\min(p,k)=p$ and $Q=\min(q,k)=q$. Representations satisfying (6.16) are said to be *highly standard* since (6.16) is a stronger constraint than (6.6).

Once again whether or not $\{k(\bar{\nu};\mu)\}$ is highly standard for U(p, q) the coefficients appearing in (6.9) can, on comparison with (6.10), be seen to be identical with those appearing in the branching rule from U$(k)\times$U(k) to U(k)

$$\{\bar{\tau}\}\times\{\sigma\}\to\sum_{\nu,\mu} R^{\bar{\tau};\sigma}_{\bar{\nu};\mu}\{\bar{\nu};\mu\} \tag{6.17}$$

where of course it may be necessary to use the modification rules in U(k) to evaluate the right-hand side. Whilst useful for checking results, this does not give the required branching from U(p, q) to U$(q)\times$U(p)

$$\{k(\bar{\nu};\mu)\}\to\sum_{\tau,\sigma} R^{\bar{\tau};\sigma}_{\bar{\nu};\mu}(0\times\varepsilon^k)\cdot\{\bar{\tau}\}\times\{\sigma\}. \tag{6.18}$$

Returning to the case (6.15) covered by the constraint (6.16) it is clear that once more there exist equivalences between highly standard harmonic series unirrep $\{k(\bar{\nu};,\mu)\}$ for various k, (ν) and (μ). In fact (6.15) gives

$$\{k(\bar{\nu};\mu)\}\to\sum_{\zeta}\{\bar{\nu}\}\cdot\{\bar{\zeta}\}\times\{\kappa\}\cdot\{\zeta\} \tag{6.19}$$

with

$$(\kappa)=(\kappa_1\kappa_2\ldots\kappa_p)=(\mu_1+k,\mu_2+k,\ldots,\mu_p+k). \tag{6.20}$$

For example in the case of U(4, 1) we have the equivalence

$$\{6(\bar{2};1)\}=\{5(\bar{2};21^3)\} \tag{6.21}$$

with both branching from U(4, 1) to U(1)$\times$U(4) to yield

$$\begin{aligned}(0\times\varepsilon^6)&\cdot\sum_{\zeta}\{\bar{2}\}\cdot\{\bar{\zeta}\}\times\{1\}\cdot\{\zeta\}\\ &=(0\times\varepsilon^6)\cdot(\{\bar{2}\}\times\{1\}+\{\bar{3}\}\times(\{2\}+\{1^2\})+\{\bar{4}\}\times(\{3\}+\{21\})+\ldots)\\ &=\{\bar{2}\}\times\{76^3\}+\{\bar{3}\}\times\{86^3\}\times\{\bar{3}\}\times\{7^26^2\}+\{\bar{4}\}\times\{96^3\}+\{\bar{4}\}\times\{876^2\}+\ldots\\ &=\sum_{\zeta}\{\bar{2}\}\cdot\{\bar{\zeta}\}\times\{76^3\}\times\{\zeta\}.\end{aligned} \tag{6.22}$$

Notice that (6.21) cannot be extended to include $\{4(\bar{2};32^3)\}$ since this leads to a violation of (6.16) and indeed of (6.6a).

The final expression (6.22) indicates on comparison with (3.12a) that

$$\{\{\bar{2};76^3\}\}=\{6(\bar{2};1)\}.$$

More generally a comparison of (6.15) with (3.10a) allows holomorphic discrete series unirreps of U(p, q) to be identified with corresponding highly standard harmonic series unirreps. To be precise

$$\begin{aligned}\{\{\bar{\nu};\kappa\}\}&=\{\{\overline{\nu_1\nu_2\ldots\nu_q};\kappa_1\kappa_2\ldots\kappa_p\}\}\\ &=\{\kappa_p(\overline{\nu_1\nu_2\ldots\nu_q};\kappa_1-\kappa_p,\kappa_2-\kappa_p,\ldots,\kappa_{p-1}-\kappa_p,0\}\}\\ &=\{\kappa(\bar{\nu};\mu)\}\end{aligned}$$

with $k=\kappa_p$ and $\mu_1=\kappa_1-\kappa_p$, $\mu_2=\kappa_2-\kappa_p,\ldots,\mu_p=0$ provided that $k=\kappa_p\geqslant p+\tilde{\nu}_1$ or $k=\kappa_p\geqslant q+\tilde{\mu}_1$.

7. Tensor products of holomorphic discrete series unirreps

Surprisingly little work seems to have been carried out on the decomposition of tensor products of holomorphic discrete series unirreps. However relatively recently the general problem has been discussed by Repka (1979) and Gutkin (1979). It is clear from the work of Repka that in analysing the tensor products of unirreps from the D_+ series of a non-compact group G a knowledge of the branching to the maximal compact subgroup K is all that is required. Indeed in such cases these branchings take the form

$$\mu \to \sum_{\sigma} B^{\sigma}_{\mu}\sigma \qquad \nu \to \sum_{\tau} B^{\tau}_{\nu}\tau \tag{7.1a, b}$$

and the tensor product decomposition for the compact group K gives

$$\sigma \times \tau = \sum_{\rho} K^{\rho}_{\sigma\tau}\rho. \tag{7.2}$$

Then clearly the tensor product representation $\mu \times \nu$ of G yields the representations of K in accordance with

$$\mu \times \nu \to \sum_{\sigma,\tau,\rho} B^{\sigma}_{\mu} B^{\tau}_{\nu} K^{\sigma}_{\sigma\tau}\rho. \tag{7.3}$$

This result enables the tensor product $\mu \times \nu$ to be decomposed in the case of holomorphic unirreps since these are necessarily labelled by their lowest K dominant weights. As pointed out implicitly by Repka (1979) this means that a formal inverse of the infinite triangular matrix B exists. One then obtains the formula for tensor products of holomorphic unirreps of the non-compact group G:

$$\mu \times \nu = \sum_{\sigma,\tau,\rho,\lambda} B^{\sigma}_{\mu} B^{\tau}_{\nu} K^{\rho}_{\sigma\tau} B^{-1\lambda}_{\rho}\lambda. \tag{7.4}$$

For example in the case of the unirreps $\langle\{2^2\}\rangle$ and $\langle\{31\}\rangle$ of Sp(4, R) we have from (5.15)

$\mathrm{Sp}(4, R) \to \mathrm{U}(2)$

$$\langle\{2^2\}\rangle \to \{2^2\} \cdot D = \{2^2\} + \{42\} + \{4^2\} + \{62\} + \{64\} + \{6^2\} + \ldots \tag{7.5a}$$

$$\langle\{31\}\rangle \to \{31\} \cdot D = \{31\} + \{3^2\} + \{42\} + \{51\} + 2\{53\} + \{5^2\} + \ldots. \tag{7.5b}$$

Thus

$\mathrm{Sp}(4, R) \times \mathrm{Sp}(4, R) \to \mathrm{U}(2)$

$$\begin{aligned}\langle\{2^2\}\rangle \times \langle\{31\}\rangle &\to (\{2^2\} + \{42\} + \{62\} + \{4^2\} + \ldots)(\{31\} + \{51\} + \{42\} + \{3^2\} + \{71\} + \ldots) \\ &= \{53\} + 2\{5^2\} + 2\{64\} + \{6^2\} + 2\{73\} + 7\{75\} + 6\{7^2\} + \ldots.\end{aligned} \tag{7.6}$$

But

$$\langle\{53\}\rangle \to \{53\} \cdot D = \{53\} + \{5^2\} + \{64\} + \{73\} + 2\{75\} + \{7^2\} + \ldots$$

$$\langle\{5^2\}\rangle \to \{5^2\} \cdot D = \{5^2\} + \{75\} + \{7^2\} + \ldots$$

$$\langle\{64\}\rangle \to \{64\} \cdot D = \{64\} + \{6^2\} + \{75\} + \ldots$$

$$\langle\{73\}\rangle \to \{73\} \cdot D = \{73\} + \{75\} + \{7^2\} + \ldots$$

$$\langle\{75\}\rangle \to \{75\}\cdot D = \{75\}+\{7^2\}+\ldots$$

$$\langle\{7^2\}\rangle \to \{7^2\}\cdot D = \{7^2\}+\ldots.$$

Hence it can be seen that the Sp(4, R) tensor product decomposes in accordance with the formula

$$\langle\{2^2\}\rangle\langle\{31\}\rangle = \langle\{53\}\rangle+\langle\{5^2\}\rangle+\langle\{64\}\rangle+\langle\{73\}\rangle+2\langle\{75\}\rangle+\langle\{7^2\}\rangle+\ldots. \tag{7.7}$$

However this result can be obtained much more readily by noting that for such holomorphic discrete series unirreps

$$\langle\{\mu\}\rangle \to \{\mu\}\cdot D \qquad \langle\{\nu\}\rangle \to \{\nu\}\cdot D \tag{7.8a, b}$$

so that

$$\langle\{\mu\}\times\langle\{\nu\}\rangle \to \{\mu\}\cdot\{\nu\}\cdot D\cdot D. \tag{7.9}$$

This leads directly to the result

$$\langle\{\mu\}\rangle\times\langle\{\nu\}\rangle = \langle\{\mu\cdot\nu\cdot D\}\rangle \tag{7.10}$$

where in (7.8)-(7.10) the symbol $\cdot$ means the tensor product appropriate to the group U(n). All terms labelled by partitions involving more than n parts are discarded.

By using precisely the same argument it follows from (6.19) that tensor products of holomorphic discrete series of unirreps of U(p, q) may be decomposed into a sum of unirreps through the use of the formula

$$\{\{\bar{\nu};\mu\}\}\times\{\{\bar{\tau};\sigma\}\} = \sum_{\zeta}\{\{\bar{\nu}\cdot\bar{\tau}\cdot\bar{\zeta};\mu\cdot\sigma\cdot\zeta\}\} \tag{7.11}$$

where the products to the left and right of ; are evaluated in the groups U(q) and U(p) respectively.

Similarly for SO*($2n$)

$$[\{\mu\}]\times[\{\nu\}] = [\{\mu\cdot\nu\cdot B\}] \tag{7.12}$$

with the products evaluated in U(n) on the right-hand side.

The justification for these extremely simple results (7.10)-(7.12) lies in the fact that the tensor products of holomorphic discrete series unirreps associated with the D_+ series can always be decomposed into a direct sum of holomorphic discrete series unirreps also from D_+. To see this it is merely necessary to note that the conditions (3.9a), (3.9b) and (3.9c) are automatically satisfied by each term in the right-hand side of (7.10), (7.11) and (7.12) respectively.

For example in (7.10), by definition $\mu_n > n$ and $\nu_n > n$ so that every term $\{\lambda\}$ appearing in $\{\mu\}\cdot\{\nu\}\cdot D$ evaluated in U(n) necessarily has $\lambda_n > 2n > n$ as required.

These results (7.10), (7.11) and (7.12) are the realisation in terms of Schur functions of the general formula appropriate to tensor products of holomorphic discrete series unirreps derived by Gutkin (1979). Casting it in terms of Schur functions allows tensor products to be evaluated very rapidly by making use of the Littlewood-Richardson rule (Littlewood 1940, p 94) either by hand or better still by making use of a computer program such as SCHUR (Black 1983).

The results for the holomorphic discrete series unirreps apply to those members of the harmonic series unirreps which are highly standard in the sense of (5.10). Thus in the case of Sp($2n$, R) we have

$$\langle\tfrac{1}{2}k(\mu)\rangle\times\langle\tfrac{1}{2}l(\nu)\rangle = \langle\tfrac{1}{2}(k+l)(\{\mu\cdot\nu\cdot D\}\rangle \tag{7.13}$$

provided that

$$\tilde{\mu}_1 \leqslant n \leqslant k - \tilde{\mu}_2 \qquad \text{and} \qquad \tilde{\nu}_1 \leqslant n \leqslant l - \tilde{\nu}_2. \tag{7.14}$$

For example in the case of Sp(4, R)

$$\begin{aligned}\langle 1(1^2)\rangle \times \langle 1(2)\rangle &= \langle 2(\{1^2\} \cdot \{2\} \cdot D)\rangle \\ &= \langle 2(\{1^2\} \cdot \{2\} \cdot (\{0\} + \{2\} + \{2^2\} + \{4\} + \{6\} + \ldots))\rangle \\ &= \langle 2(\{31\} + \{3^2\} + \{42\} + \{51\} + 2\{53\} + \{5^2\} + \ldots .\end{aligned} \tag{7.15}$$

This is in accord, as it must be, with (7.7).

It should be noted that in this example as in all others the cut-off with respect to n in $\{\mu\} \cdot \{\nu\} \cdot D$ along with the constraints (7.14) ensure that each term $\{\rho\}$ in the product is automatically highly standard. To see this note that

$$n \geqslant \tilde{\rho}_1 \geqslant \tilde{\rho}_2 \geqslant \ldots$$

so that

$$n \geqslant \tilde{\rho}_1 \qquad \text{and} \qquad \tilde{\rho}_1 + \tilde{\rho}_2 \leqslant 2n \leqslant k - \tilde{\mu}_2 + l - \tilde{\nu}_2 \leqslant k + l$$

giving

$$\tilde{\rho}_1 \leqslant n \leqslant k + l - \tilde{\rho}_2$$

as required for a highly standard unirrep.

Similarly for U(p, q)

$$\{k(\bar{\nu}; \mu)\} \times \{l(\bar{\tau}; \sigma)\} = \sum_{\zeta} \{k + l(\bar{\nu} \cdot \bar{\tau} \cdot \bar{\zeta}; \mu \cdot \sigma \cdot \zeta)\} \tag{7.16}$$

provided that

$$\tilde{\mu}_1 \leqslant p, \qquad \tilde{\nu}_1 \leqslant q, \qquad \tilde{\sigma}_1 \leqslant p, \qquad \tilde{\tau}_1 \leqslant q$$

and either $p \leqslant k - \tilde{\nu}_1$ or $q \leqslant k - \tilde{\mu}_1$ and either $p \leqslant k - \tilde{\tau}_1$ or $q \leqslant k - \tilde{\sigma}_1$.

8. Tensor products of harmonic unirreps of Sp(2*n*, *R*)

The real difficulties in evaluating tensor products of the unirreps considered in this paper only occur when dealing with those harmonic series unirreps which are not holomorphic discrete series unirreps.

In considering such unirreps of Sp($2n$, R) for example the difficulty lies in the fact that if the highly standard condition (5.10) is violated, then the very simple branching rule (5.9) for the restriction to U(n) must be replaced by the more complicated rule (5.6). Thus using

$$\langle \tfrac{1}{2}k(\mu)\rangle = \varepsilon^{k/2}\{\{\mu_s\}^k_M \cdot D_M\}_M \qquad \text{with } M = \min(n, k) \tag{8.1a}$$

$$\langle \tfrac{1}{2}l(\mu)\rangle = \varepsilon^{l/2}\{\{\nu_s\}^k_N \cdot D_N\}_N \qquad \text{with } N = \min(n, l) \tag{8.1b}$$

gives

$$\langle \tfrac{1}{2}k(\mu)\rangle \times \langle \tfrac{1}{2}l(\nu)\rangle \to \varepsilon^{(k+l)/2}\{\{\{\mu_s\}^k_M \cdot D_M\}_M \cdot \{\{\nu_s\}^l_N \cdot D_N\}_N\}_n \tag{8.2}$$

which cannot be trivially inverted to give terms $\langle \frac{1}{2}(k+l)(\lambda)\rangle$ because of the signed sequences that appear in this expression and the complicated dependence on the relative values of n, k and l. However the inversion may be carried out step by step using the fact that the harmonic series unirreps are labelled by the highest weight of their lowest U(n) constituent representation.

For example in the case of Sp(8, R) branching to U(4)

$$\begin{aligned}\langle 1(1^2)\rangle &\to \varepsilon\{\{1^2{}_s\}_2^2 \cdot D_2\}_2 \\ &= \varepsilon\{\{1^2\} \cdot (\{0\}+\{2\}+\{2^2\}+\{4\}+\ldots)\}_2 \\ &= \varepsilon(\{1^2\}+\{31\}+\{3^2\}+\{51\}+\{53\}+\{5^2\}+\ldots)\end{aligned} \tag{8.3}$$

and

$$\begin{aligned}\langle 1(2)\rangle &\to \varepsilon\{\{2_s\}_2^2 \cdot D_2\}_2 \\ &= \varepsilon\{(\{2\}-\{2^2\}) \cdot (\{0\}+\{2\}+(2^2\}+\{4\}+\ldots)\}_2 \\ &= \varepsilon(\{2\}+\{31\}+\{4\}+\{42\}+\{51\}+\{53\}+\ldots).\end{aligned} \tag{8.4}$$

These give

$$\langle 1(1^2)\rangle \times \langle 1(2)\rangle \to \varepsilon^2(\{21^2\}+\{31\}+\{31^3\}+2\{321\}+\{3^2\}+\{3^21^2\}+2\{3^22\}+\ldots). \tag{8.5}$$

The leading term $\varepsilon^2\{21^2\}$ arises from the branching

$$\langle 2(21^2)\rangle \to \varepsilon^2\{\{21^2_s\}_4^4 \cdot D_4\}_4$$

$$= \varepsilon^2\{(\{21^2\}-\{2^21^2\})_4 \cdot (\{0\}+\{2\}+\{2^2\}+\{4\}+\ldots)_4\}_4 \tag{8.6}$$

$$= \varepsilon^2(\{21^2\}+\{31^3\}+\{321\}+\{3^21^2\}+\{3^22\}+\{3^31\}+\ldots). \tag{8.7}$$

Subtracting these terms from (8.5) leaves the next leading term $\varepsilon^2\{31\}$ associated with the branching

$$\begin{aligned}\langle 2(31)\rangle &\to \varepsilon^2\{\{31_s\}_4^4 \cdot D_4\}_4 \\ &= \varepsilon^2\{(\{31\}-\{32^21\})_4 \cdot (\{0\}+\{2\}+\{2^2\}+\{4\}+\ldots) \\ &= \varepsilon^2(\{31\}+\{321\}+\{3^2\}+\{3^22\}+\ldots).\end{aligned} \tag{8.8}$$

This may be subtracted and the next leading term $\langle 2(41^2)\rangle$ identified.

Painfully one can proceed to the final result

$$\begin{aligned}\langle 1(1^2)\rangle \times \langle 1(2)\rangle \\ &= \langle 2(21^2)\rangle + \langle 2(31)\rangle + \langle 2(41^2)\rangle + \langle 2(42)\rangle + \langle 2(51)\rangle + \langle 2(53)\rangle \\ &\quad + \langle 2(61^2)\rangle + \langle 2(62)\rangle + \langle 2(64)\rangle + \ldots.\end{aligned} \tag{8.9}$$

Clearly the aim should be to circumvent the intricacies and indeed overcounting of this particular technique. To this end it is worth considering following a suggestion of Rowe (1984) the group-subgroup chains

$$\begin{aligned}\mathrm{Sp}(2nk+2nl,\ R) &\to \mathrm{Sp}(2nk,\ R) \times \mathrm{Sp}(2nl,\ R) \\ &\to \mathrm{Sp}(2n,\ R) \times \mathrm{O}(k) \times \mathrm{Sp}(2n,\ R) \times \mathrm{O}(l) \\ &\to \mathrm{Sp}(2n,\ R) \times \mathrm{O}(k) \times \mathrm{O}(l)\end{aligned} \tag{8.10}$$

and

$$\begin{aligned}\mathrm{Sp}(2nk+2nl,\ R) &\to \mathrm{Sp}(2n,\ R) \times \mathrm{O}(k+l) \\ &\to \mathrm{Sp}(2n,\ R) \times \mathrm{O}(k) \times \mathrm{O}(l).\end{aligned} \tag{8.11}$$

The corresponding branching rules are

$$\tilde{\Delta} \to \tilde{\Delta} \times \tilde{\Delta} \to \sum_{\mu,\nu} \langle \tfrac{1}{2}k(\mu)\rangle \times [\mu] \times \langle \tfrac{1}{2}l(\nu)\rangle \times [\nu]$$

$$\to \sum_{\mu,\nu} \langle \tfrac{1}{2}k(\mu)\rangle \times \langle \tfrac{1}{2}l(\nu)\rangle \times [\mu] \times [\nu] \tag{8.12}$$

and

$$\Delta \to \sum_{\lambda} \langle \tfrac{1}{2}(k+l)(\lambda)\rangle \times [\lambda]$$

$$\to \sum_{\lambda} \langle \tfrac{1}{2}(k+l)(\lambda)\rangle \times \sum_{\mu,\nu} R_{\lambda}^{\mu\nu} [\mu] \times [\nu] \tag{8.13}$$

where the coefficients $R_{\lambda}^{\mu\nu}$ are the branching rule coefficients appropriate to the restriction from $O(k+l)$ to $O(k) \times O(l)$, i.e.

$$[\lambda] \to \sum_{\mu,\nu} R_{\lambda}^{\mu\nu} [\mu] \times [\nu]. \tag{8.14}$$

The linear independence of the characters $[\mu] \times [\nu]$ of $O(k) \times O(l)$ then ensures, on comparison with (8.12) and (8.13), that

$$\langle \tfrac{1}{2}k(\mu)\rangle \times \langle \tfrac{1}{2}l(\nu)\rangle = \sum_{\lambda} R_{\lambda}^{\mu\nu} \langle \tfrac{1}{2}(k+l)(\lambda)\rangle \tag{8.15}$$

for tensor products of $Sp(2n, R)$ harmonic series unirreps, whose coefficients are precisely the branching rule coefficients of (8.14).

This result (8.15) is not unlike that of (5.14) and shares similar disadvantages in that the direct evaluation of (8.15) from the use of (6.14) involves a summation over λ rather than μ and ν and the derivation of the coefficients in (8.14) involves in general the use of modification rules.

Indeed from the usual Schur function techniques (King 1975) for evaluating these branchings one obtains a more explicit statement of (8.13)

$$\tilde{\Delta} \to \sum_{\lambda} \langle \tfrac{1}{2}(k+l)(\lambda)\rangle \times [\lambda]$$

$$\to \sum_{\lambda,\rho} \langle \tfrac{1}{2}(k+l)(\lambda)\rangle \times [\rho] \times [\lambda/\rho \cdot D] \tag{8.16a}$$

$$= \sum_{\rho,\kappa} \langle \tfrac{1}{2}(k+l)(\{\rho\} \cdot \{\kappa\} \cdot D)\rangle \times [\rho] \times [\kappa] \tag{8.16b}$$

$$= \sum_{\mu,\nu} \langle \tfrac{1}{2}(k+l)(\{\{\mu_s\}^k \cdot \{\nu_s\}^l \cdot D\})\rangle \times [\mu] \times [\nu] \tag{8.16c}$$

where in (8.16*a*), (λ) is necessarily restricted by the conditions (4.4) which here take the form:

$$\tilde{\lambda}_1 \leqslant n \qquad \text{and} \qquad \tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant k+l. \tag{8.17}$$

This imposes limitations first on (ρ) and then (κ) and the allowed terms (δ) of D in (8.16*b*). However the terms $[\rho]$ and $[\kappa]$ must then be modified to give standard labels $[\mu]$ and $[\nu]$ of $O(k)$ and $O(l)$, respectively, with

$$\tilde{\mu}_1 \leqslant n \qquad \text{and} \qquad \tilde{\mu}_1 + \tilde{\mu}_2 \leqslant k$$

$$\tilde{\nu}_1 \leqslant n \qquad \text{and} \qquad \tilde{\nu}_1 + \tilde{\nu}_2 \leqslant l$$

as appropriate to (8.12). This then accounts for the need to introduce the signed sequences $\{\mu_s\}^k$ and $\{\nu_s\}^l$ in (8.16*c*). The final point to notice is that the restriction (8.17) must apply to all the terms $\langle\frac{1}{2}(k+l)(\lambda)\rangle$ of (8.16*c*). All other terms arising in the product $\{\mu_s\}^k\cdot\{\nu_s\}^l\cdot D$ are to be discarded. With this important proviso it then follows from (8.13) and (8.16*c*) that for tensor products of Sp(2*n*, *R*)

$$\langle\tfrac{1}{2}k(\mu)\rangle\times\langle\tfrac{1}{2}l(\nu)\rangle=\langle\tfrac{1}{2}(k+l)((\{\mu_s\}^k\cdot\{\nu_s\}^l\cdot D))_{k+l,n}\rangle \tag{8.18}$$

where $((\lambda))_{k+l,n}$ is to be interpreted as null unless (4.4) is satisfied i.e.

$$((\lambda))_{k+l,n}=\begin{cases}(\lambda) & \text{if } \tilde\lambda_1\leqslant n \text{ and } \tilde\lambda_1+\tilde\lambda_2\leqslant k+l\\ 0 & \text{otherwise.}\end{cases} \tag{8.19}$$

For the product (8.9) this formula gives in the case of Sp(8, *R*) the result

$$\begin{aligned}\langle 1(1^2)\rangle\times\langle 1(2)\rangle\\ &=\langle 2((\{1_s^2\}^2\cdot\{2_s\}^2\cdot D))_{4,4}\\ &=\langle 2(((\{1^2\}-\{21^2\})\cdot(\{2\}-\{2^2\})(\{0\}+\{2\}+\{2^2\}+\{4\}+(42\}+\{4^2\}+\ldots)))_{4,4}\rangle\\ &=\langle 2((\{21^2\}-\{2^21^2\}+\{31\}-\{32^21\}+\{41^2\}+\{42\}+\ldots))_{4,4}\rangle\\ &=\langle 2(21^2)\rangle+\langle 2(31)\rangle+\langle 2(41^2)\rangle+\langle 2(42)\rangle\ldots\end{aligned} \tag{8.20}$$

where (8.19) has been used to eliminate terms such as $\{31^3\}$ and $\{3^21^2\}$ appearing in $\{1_s^2\}^2$ and $\{2_s^2\}^2$ as well as others appearing in the final product. The final result is in agreement with (8.9).

There is of course a remarkable similarity between the formula (8.16) for tensor products of harmonic series unirreps and the simpler formula (7.10) for tensor products of holomorphic discrete series unirreps. Indeed (7.10) is a special case of (8.16) which applies if $\langle\frac{1}{2}k(\mu)\rangle$ and $\langle\frac{1}{2}l(\nu)\rangle$ are both highly standard.

Whilst (8.16) is certainly easier to use than the technique based on (8.2) it may still involve considerable overcounting by virtue of the need to evaluate two signed sequences before multiplying by *D* and truncating via (8.17).

The consideration of numerous examples lead us to *conjecture* the validity of the following somewhat simpler formula

$$\begin{aligned}\langle\tfrac{1}{2}k(\mu)\times\langle\tfrac{1}{2}l(\nu)\rangle\\ &=\langle\tfrac{1}{2}(k+l)(\{\mu\}\cdot\{\{\nu_s\}_N^l\cdot D_N\})\rangle_n \qquad \text{with } N=\min(n,l) \qquad (8.21a)\\ &=\langle\tfrac{1}{2}(k+l)(\{\nu\}\cdot\{\{\mu_s\}_M^k\cdot D_M\}_M)\rangle_n \qquad \text{with } M=\min(n,k). \qquad (8.21b)\end{aligned}$$

The extra degree of simplification lies in the occurrence, in each of these alternative expressions, of a single signed sequence. This time however the symbol $\langle\frac{1}{2}(k+l)(\lambda)\rangle_n$ is to be interpreted as a harmonic series unirrep of Sp(2*n*, *R*). The modification converts it to standard form by means of a two stage procedure: first modification in O(*k*+*l*) and then modification in U(*n*). Using these modification rules, we have for example

$$\langle 2(321)\rangle_n=0 \qquad \text{for all } n \tag{8.22a}$$

$$\langle 2(3^22)\rangle_n=\begin{cases}-\langle 2(3^2)\rangle & \text{for } n\geqslant 2\\ 0 & \text{for } n=1\end{cases} \tag{8.22b}$$

$$\langle 2(621^2)\rangle_n=\begin{cases}-\langle 2(61^2)\rangle & \text{for } n\geqslant 3\\ 0 & \text{for } n=1,2\end{cases} \tag{8.22c}$$

where for (8.22*b*) it should be noted that

$$[3^2]=[3^2]^* \qquad \text{in O(4)}$$

whilst for (8.22*c*) use has been made of the identity

$$[61^2]=[6]^* \qquad \text{in O(4)}.$$

As an exemplification of (8.21*b*), we have

$$\begin{aligned}
&\langle 1(1^2)\rangle\times\langle 1(2)\rangle\\
&\quad=\langle 2(\{2\}\cdot(\{1^2\}+\{31\}+\{3^2\}+\{51\}+\{53\}+\{52\}+\ldots))\rangle_4\\
&\quad=\langle 2(21^2+31+321+3^2+3^22+41^2+42+431+51+521+2(53)\\
&\qquad+532+541+5^2+5^22+\ldots)\rangle_4\\
&\quad=\langle 2(21^2)\rangle+\langle 2(31)\rangle+\langle 2(41^2)\rangle+\langle 2(42)\rangle+\langle 2(51)\rangle+\langle 2(53)\rangle+\ldots
\end{aligned}$$

in agreement with (8.9). Use has been made of the modifications

$$\begin{aligned}
&\langle 2(321)\rangle_4=\langle 2(431)\rangle_4=\langle 2(521)\rangle_4=\langle 2(541)\rangle_4=0\\
&\langle 2(3^22)\rangle_4=-\langle 2(3^2)\rangle\\
&\langle 2(532)\rangle_4=-\langle 2(53)\rangle\\
&\langle 2(5^22)\rangle_4=-\langle 2(5^2)\rangle.
\end{aligned}$$

Similarly (8.21*a*) gives

$$\begin{aligned}
&\langle 1(1^2)\rangle\times\langle 1(2)\rangle\\
&\quad=\langle 2\{1^2\}\cdot(\{2\}-\{2^2\})\cdot D)\rangle_4\\
&\quad=\langle 2(\{1^2\}\cdot(\{2\}+\{31\}+\{4\}+\{42\}+\{51\}+\{53\}+\ldots))\rangle_4\\
&\quad=\langle 2(21^2+31+31^3+321+2(41^2)+42+421^2+431+31^3+\ldots)\rangle_4\\
&\quad=\langle 2(21^2)\rangle+\langle 2(31)\rangle+\langle 2(41^2)\rangle+\langle 2(42)\rangle+\ldots
\end{aligned}$$

thanks to the identities

$$\begin{aligned}
&\langle 2(31^3)\rangle_4=\langle 2(321)\rangle_4=\langle 2(431)\rangle_4=0\\
&\langle 2(421^2)\rangle_4=-\langle 2(41^2)\rangle_4.
\end{aligned}$$

The structure of the formulae (8.21*a*) and (8.21*b*) is worth some comment. They may be recast in the form

$$\langle\tfrac{1}{2}k(\mu)\rangle\times\langle\tfrac{1}{2}l(\nu)\rangle=\left\langle\tfrac{1}{2}(k+l)\left(\sum_{\tau}R_{\nu}^{\tau}\{\tau\}\cdot\{\mu\}\right)\right\rangle_n \tag{8.23a}$$

$$=\left\langle\tfrac{1}{2}(k+l)\left(\sum_{\sigma}R_{\mu}^{\sigma}\{\sigma\}\cdot\{\nu\}\right)\right\rangle_n. \tag{8.23b}$$

The notation of (5.13) has been used in which the coefficients R_{μ}^{σ} and R_{ν}^{τ} are determined by the branching rules for the restrictions from U(k) to O(k) and from U(l) to O(l) respectively. In this form they are completely analogous to the recently derived rule (Black *et al* 1983) for evaluating tensor products of unirreps of the orthogonal group

O(2n) or O(2n+1) which take the form

$$[\mu]\times[\nu]=\sum_{\tau} R^{\tau}_{\nu}[\{\tau\}\cdot\{\mu\}] \tag{8.24a}$$

$$=\sum_{\sigma} R^{\sigma}_{\mu}[\{\sigma\}\cdot\{\nu\}] \tag{8.24b}$$

where the products signified by · are to be evaluated in U(n), with the final expression modified in O($2n$) or O($2n+1$) as appropriate. It was in fact this analogy which led to the consideration of the conjecture (8.21). The basis of (8.24) lies in the fact that if $\boldsymbol{\rho}$ denotes half the sum of the positive roots of O($2n$) or O($2n+1$) and $\boldsymbol{\rho}^{\mathrm{c}}$ half the sum of the positive roots of U(n) then $\boldsymbol{\rho}-\boldsymbol{\rho}^{\mathrm{c}}=\boldsymbol{\delta}$ where $\boldsymbol{\delta}$ is either $(0,0,\ldots,0)$ or $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$. The derivation depends only upon $\boldsymbol{\delta}$ being of the form $(\delta,\delta,\ldots,\delta)$ however. The fact that $\boldsymbol{\rho}-\boldsymbol{\rho}^{\mathrm{c}}=\boldsymbol{\rho}^{\mathrm{n}}$ is precisely of this form in (3.4b) supports the notion that the analogy between (8.23) and (8.24) may be exact save for the modification rules appropriate to each case.

Further clues as to the validity of the conjecture (8.21) present themselves through a comparison with the result (8.19) in the case for which either $\langle\frac{1}{2}k(\mu)\rangle$ or $\langle\frac{1}{2}l(\nu)\rangle$ is highly standard, and with the known result

$$\langle\tfrac{1}{2}k(\mu)\rangle\times\langle\tfrac{1}{2}l(\nu)\rangle=\langle\tfrac{1}{2}(k+l)(\{\mu\}\cdot\{\nu\}\cdot D)\rangle \tag{8.25}$$

based on (7.10) if both unirreps are highly standard.

Unfortunately we have not been able to establish a rigorous proof of (8.21).

One further piece of evidence in support of (8.21) can be found by checking this rule against the known decomposition (Kashiwara and Vergne 1978) of powers of the basic harmonic representation $\tilde{\Delta}$. Under the restriction from Sp($2n$, R) to U(n)

$$\tilde{\Delta}=\langle\tfrac{1}{2}(0)\rangle+\langle\tfrac{1}{2}(1)\rangle\to\varepsilon^{1/2}M \tag{8.26a}$$

with

$$M=\{0\}+\{1\}+\{2\}+\{3\}+\ldots.$$

From (8.21) it follows that

$$\begin{aligned}\tilde{\Delta}^2=\tilde{\Delta}\times\tilde{\Delta}&=\langle 1((\{0\}+\{1\})\cdot M)\rangle_n\\&=\langle 1(\{0\}+\{1\}+\{2\}+\{3\}+\ldots+\{1\}+\{2\}+\{1^2\}\\&\quad+\{3\}+\{21\}+\{4\}+\{51\}+\ldots)\rangle_n\\&=\langle 2(\{0\}+\{1^2\}+2\{2\}+2\{3\}+2\{4\}+\ldots)\rangle\end{aligned} \tag{8.26b}$$

where use has been made of the fact that

$$\langle 1(m1)\rangle_n=0 \qquad \text{for } m>1.$$

The expressions (8.26a) and (8.26b) take the form

$$\tilde{\Delta}^1=\sum_{\lambda} d_1[\lambda]\langle\tfrac{1}{2}(\lambda)\rangle \qquad \tilde{\Delta}^2=\sum_{\lambda} d_2[\lambda]\langle 1(\lambda)\rangle \tag{8.27a, b}$$

as can be seen by evaluating the dimensions $d_k[\lambda]$ of the unirrep $[\lambda]$ of O(k). The generalisation of this result which suggests itself and which was derived by Kashiwara and Vergne (1978) is

$$\tilde{\Delta}^k=\sum_{\lambda} d_k[\lambda]\langle\tfrac{1}{2}k(\lambda)\rangle \tag{8.28}$$

with the summation over all standard labels satisfying (4.4). This follows by induction from (8.16) and (8.18) and the case of (8.28) with k replaced by $k-1$. To see this, note that

$$\tilde{\Delta}^k = \tilde{\Delta}^{k-1} \times \tilde{\Delta} = \sum_{\mu} d_{k-1}[\mu]\langle \tfrac{1}{2}(k-1)(\mu)\rangle \times (\langle \tfrac{1}{2}(0)\rangle + \langle \tfrac{1}{2}(1)\rangle)$$
$$= \sum_{\mu} d_{k-1}[\mu]\langle \tfrac{1}{2}k(\{\mu\} \cdot M)\rangle_n$$
$$= \sum_{\lambda} d_{k-1}[\lambda/M]\langle \tfrac{1}{2}k(\lambda)\rangle_n \qquad (8.29)$$

from (8.21) and (8.26). By hypothesis $[\mu]$ is standard in $\mathrm{O}(k-1)$ and if k is even multiplication by M can lead only to terms $[\lambda]$ standard in $\mathrm{O}(k)$, whilst for k odd non-standard terms of $\mathrm{O}(k)$ may also arise which are subject to modification rules. These rules are precisely what are required to make all these non-standard contributions vanish identically. However it is well known that under the restriction from $\mathrm{O}(k)$ to $\mathrm{O}(k-1)$

$$[\lambda] \to [\lambda/M] \qquad (8.30)$$

so

$$d_k[\lambda] = d_{k-1}[\lambda/M]. \qquad (8.31)$$

This completes a very simple inductive 'derivation' of (8.28) based on the conjecture (8.21).

The particular procedure followed in deriving (8.28) from (8.21) shows one very important facet of (8.21), namely the fact that although individually the formulae look asymmetric in (μ) and (ν), as a pair they are symmetrical. Which of (8.21a) and (8.21b) is to be used in a particular case depends upon the relative values of k and l and on the nature of (μ) and (ν). That there is a choice is very advantageous in some cases.

9. Tensor products of harmonic unirreps of U(p, q)

Turning to $\mathrm{U}(p, q)$ the counterpart of the chains (8.10) and (8.11) are

$$\mathrm{U}(pk+pl, qk+ql) \to \mathrm{U}(pk, qk) \times \mathrm{U}(pl, ql)$$
$$\to \mathrm{U}(p, q) \times \mathrm{U}(k) \times \mathrm{U}(p, k) \times \mathrm{U}(l)$$
$$\to \mathrm{U}(p, q) \times \mathrm{U}(k) \times \mathrm{U}(l) \qquad (9.1)$$

and

$$\mathrm{U}(pk+pl, qk+ql) \to \mathrm{U}(p, q) \times \mathrm{U}(k+l)$$
$$\to \mathrm{U}(p, q) \times \mathrm{U}(k) \times \mathrm{U}(l) \qquad (9.2)$$

with branching rules

$$H \to H \times H \to \sum_{\mu,\nu,\sigma,\tau} (\{k(\bar{\nu}; \mu)\} \times \{\bar{\nu}; \mu\} \times \{l(\bar{\tau}; \sigma)\} \times \{\bar{\tau}; \sigma\}$$
$$\to \sum_{\mu,\nu,\sigma,\tau} (\{k(\bar{\nu}; \mu)\} \times \{l(\bar{\tau}; \bar{\sigma})\}) \times (\{\bar{\nu}; \mu\} \times \{\bar{\tau}; \sigma\}) \qquad (9.3)$$

and

$$H \to \sum_{\lambda,\rho} \{k+l(\bar{\rho};\lambda)\} \times \{\bar{\rho},\lambda\}$$
$$\to \sum_{\lambda,\rho} \{k+l(\bar{\rho};\lambda)\} \times \sum_{\nu,\mu,\tau,\sigma} R_{\bar{\rho};\lambda}^{\bar{\nu};\mu\ \bar{\tau};\sigma} \{\bar{\nu};\mu\} \times \{\bar{\tau};\sigma\} \tag{9.4}$$

where the coefficients $R^{\cdots}_{\cdot\cdot}$ are $\mathrm{U}(k+l) \to \mathrm{U}(k) \times \mathrm{U}(l)$ branching rule coefficients. Hence

$$\{k(\bar{\nu};\mu)\} \times \{l(\bar{\tau};\sigma)\} = \sum_{\rho,\lambda} R_{\bar{\rho};\lambda}^{\bar{\nu};\mu\ \bar{\tau};\sigma} \{k+l(\bar{\rho};\lambda)\} \tag{9.5}$$

for tensor products of $\mathrm{U}(p, q)$ harmonic series unirreps (cf (8.15) for $\mathrm{Sp}(2n, R)$).

Since for $\mathrm{U}(k+l) \supset \mathrm{U}(k) \times \mathrm{U}(l)$ it is known that (King 1975)

$$\{\bar{\rho};\lambda\} \to \sum_{\zeta,\eta,\xi} \{\overline{\rho/\xi};\lambda/\eta\} \times \{\overline{\xi/\zeta};\eta/\xi\} \tag{9.6}$$

it follows that

$$\{k(\bar{\nu};\mu)\} \times \{l(\bar{\tau};\sigma)\} = \sum_{\zeta} \{k+l((\{\bar{\nu}_{\rm s}\}^k \cdot \{\bar{\tau}_{\rm s}\}^l \cdot \{\bar{\zeta}\};\{\mu_{\rm s}\}^k \cdot \{\sigma_{\rm s}\}^l \cdot \{\zeta\}))\} \tag{9.7}$$

where it is to be understood that terms are paired together so that $\{\bar{\nu}_{\rm s};\mu_{\rm s}\}^k$ and $\{\bar{\tau}_{\rm s};\sigma_{\rm s}\}^l$ are signed sequences calculated through the use of modification rules of $\mathrm{U}(k)$ and $\mathrm{U}(l)$ respectively. As in (8.19) it is also to be understood that

$$((\bar{\rho};\lambda))_{k+l,p,q} = \begin{cases} (\bar{\rho};\lambda) & \text{if } \tilde{\lambda}_1 \leqslant p,\ \tilde{\rho}_1 \leqslant q \text{ and } \tilde{\lambda}_1 + \tilde{\rho}_1 \leqslant k+l \\ 0 & \text{otherwise.} \end{cases} \tag{9.8}$$

Once more a *conjecture* similar to (8.21) can be formulated namely

$$\{k(\bar{\nu};\mu)\} \times \{l(\bar{\tau};\sigma)\} = \sum_{\zeta} \{k+l(\{\bar{\nu}\} \cdot \{\bar{\tau}_{\rm s}\}^l_Q \cdot \{\bar{\zeta}\}_Q;\{\mu\} \cdot \{\sigma_{\rm s}\}^k_P \cdot \{\zeta\}_P)\}_{p,q} \tag{9.9}$$

with $P = \min(p, k)$ and $Q = \min(q, l)$, where the symbol $\{k+l(\bar{\rho};\lambda)\}_{p,q}$ is to be interpreted as a unirrep of $\mathrm{U}(p, q)$ only after modifying $\{\bar{\rho};\lambda\}$ first with respect to $\mathrm{U}(k+l)$ and then with respect to $\mathrm{U}(q) \times \mathrm{U}(p)$.

Just as

$$H = H_0 + \sum_{m=1}^{\infty} (H_m + H_{-m}) = \sum_{\mu,\nu} d_1\{\bar{\nu};\mu\}\{1(\bar{\nu};\mu)\}, \tag{9.10}$$

so the conjecture (9.9) can be used to generate by induction the result

$$H^k = \sum_{\mu,\nu} d_k\{\bar{\nu};\mu\}\{k(\bar{\nu};\mu)\} \tag{9.11}$$

as derived rigorously by Kashiwara and Vergne (1978). Of course (9.10) underlies the crucial formula (6.3) which serves to define the harmonic series unirreps of $\mathrm{U}(p, q)$.

10. Conclusions

The results described here demonstrate that Schur function methods have a useful role to play in studying the properties of holomorphic discrete series and harmonic series unirreps of the non-compact groups $\mathrm{Sp}(2n, R)$ and $\mathrm{U}(p, q)$.

The branching rules from these groups to the maximal compact subgroups $\mathrm{U}(n)$ and $\mathrm{U}(q) \times \mathrm{U}(p)$ respectively take the very simple forms (5.9) and (6.15) in the case of the holomorphic discrete series unirreps (5.9) and (6.15). Although the general harmonic series unirreps are slightly more complicated to deal with this can be accomplished through the use of signed sequences as in (5.6) and (6.12).

Furthermore tensor product formulae have been derived in all cases, along with the conjectures (8.21) and (9.9) which, if valid, very much simplify the evaluation of tensor products for $Sp(2n, R)$ and $U(p, q)$.

Little attention has been given to $SO^*(2n)$ but we suspect that comparable formulae can be derived in this case, it being merely necessary to change D to B and modify other rules appropriately.

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